Trumpet slices in Kruskal-like coordinates

It is possible to write the Schwarzschild metric in the Kruskal-type coordinates T, R, θ, ϕ as

$$ds^2 = F(T,R)\left(-dT^2 + dR^2\right) + r_S(T,R)^2 d\Omega^2,$$

where r_S is the Schwarzschild radial coordinate. The Kruskal's coordinates R_K, T_K

$$T_K = \frac{v_K + u_K}{2}, \quad R_K = \frac{v_K - u_K}{2}$$

$$u_K = -\sqrt{\left|\frac{r_S}{2M} - 1\right|}\, e^{\frac{r_S - t_S}{4M}} \operatorname{sgn}(r_S - 2M), \; v_K = \sqrt{\left|\frac{r_S}{2M} - 1\right|}\, e^{\frac{t_S + r_S}{4M}}$$

have inappropriate behavior farther from the horizon, so instead to cover the area $v > 0$ of the Schwarzschild manifold (see [1]) we will use

$$T = \frac{v+u}{2}, \quad R = \frac{v-u}{2}$$

$$u = -4M \operatorname{arcsinh}\left(\sqrt{\left|\frac{r_S}{2M} - 1\right|}\, e^{\frac{r_S - t_S}{4M}}\right) \operatorname{sgn}(r_S - 2M),$$

$$v = 4M \operatorname{arcsinh}\left(\sqrt{\left|\frac{r_S}{2M} - 1\right|}\, e^{\frac{r_S + t_S}{4M}}\right)$$

With these coordinates we have

$$t_s = -2M \log\left|\frac{U_K}{V_K}\right| = -2M \log\left|\frac{\sinh\frac{u}{4M}}{\sinh\frac{v}{4M}}\right|, \tag{1}$$

$$\left(\frac{r_s}{2M} - 1\right)\exp\left(\frac{r_s}{2M}\right) = -\sinh\frac{u}{4M}\sinh\frac{v}{4M}, \tag{2}$$

$$F(T,R) = -\left(1 - \frac{2M}{r_s}\right)\coth\frac{u}{4M}\coth\frac{v}{4M}. \tag{3}$$

Since at the spatial infinity $v \approx -u \approx \infty$ we get $F(T,R) \approx 1$ there and T, R behave there approximately as time and space coordinates of the Minkowski spacetime.

Problem

Consider a hypersurface Σ given by prescription $T = H(R)$ (when given in T-R coordinates) or $t_S = h(r_s)$ if we use the Schwarzschild coordinate t_S, r_S and function h used in [2] (note that there the Schwarzschild coordinates are named T and R).

- Consider the differential equation for the trumpet slice, which according to [2] reads

  $$fh' = -\frac{1}{\alpha}\frac{C_n}{r_s^2}e^{\frac{\alpha}{n}} \tag{4}$$

  where $f = 1 - 2M/r_S$, α is given in paper [2] by eqs. (37-39) and C_n by eqs. (42) and (43). The Schwarzschild coordinates are not regular at the horiozn and thus also $h(r_S)$ diverges there. It means that (4) is not suitable differential equation for unknown function $h(r_S)$.

  Nevertheless the combination fh' is regular and using the coordinate transformations given above from fh' you can compute the derivative $H' = dH(R)/dR$ and thus you can plot slice Σ in coordinates R, T, such that $T_\Sigma = H(R)$.

- Find this transformation from fh' to H' and numerically solve solve the differential equation for $H(R)$ with the boundary condition $H(R=\infty) = T_\infty$.

  According to your preferences you may choose instead of solving the transcendent algebraic equations for r_S and α to find them as solutions of respective differential equations and solve them together with the ODE for $H(R)$.

- Plot in the $R-T$ plane (in the region $v>0$) lines of constant r_S, t_S and $H(R)$ for $T_\infty/M = -4, -3, ..3, 4$ and $n=2$ (1+log slicing) and $n=\infty$ (maximal slicing).

Other information

- The derivative h' for a hypersurface given by some function $H(R)$ can be determined from the ratio

$$\frac{dh}{dr_s} = \frac{dt_s}{dr_s} = \frac{\dfrac{dt_s(R,T(R))}{dR}}{\dfrac{dr_s(R,T(R))}{dR}}. \tag{5}$$

- The derivative dr_s/dR can be obtained by differentiating the implicit function (2):

$$\frac{dr_s}{dR} = f\frac{\sinh\left(\frac{R}{2M}\right) - T'(R)\sinh\left(\frac{T(R)}{2M}\right)}{\cosh\left(\frac{R}{2M}\right) - \cosh\left(\frac{T(R)}{2M}\right)} \tag{6}$$

- Similarly

$$\frac{dt_s}{dR} = \frac{\sinh\left(\frac{R}{2M}\right)T'(R) - \sinh\left(\frac{T(R)}{2M}\right)}{\cosh\left(\frac{R}{2M}\right) - \cosh\left(\frac{T(R)}{2M}\right)} \tag{7}$$

- For the numerical solution, it is not necessary to work out the expressions up to the form when they are well defined on the horizon. The numerical method will not hit a point precisely on the horizon and the introduced numerical error does not matter for the purpose of plotting the graph.

- The asymptotic expansion of (4) allows one to find $h(r_s) \approx T_\infty + C_n e^{\frac{1}{n}} r_s^{-1} + O(r_s^{-2})$ and so we do not have to start the integration at the infinity.

In Mathematica you may use following code to get $\alpha(r_s)$:

```
αcn[n_] :=
 Sqrt[ (Sqrt[4 + 9 n^2] - 3 n)/(Sqrt[4 + 9 n^2] + 3 n)]

Cn2[n_] := Cn2[n] =
  N[(Sqrt[4 + 9 n^2] + 3 n)^3/(128 n^3) Exp[-2 αcn[n]/n] ]

αr[rs_Real, n_] := α /. FindRoot[
           α^2 == 1 - 2/rs + Cn2[n] Exp[2 α/n]/rs^4,
           {α, αcn[n] // N}   ]
```

In Python you may use following code:

```
import math
import numpy as np
import scipy.optimize
import matplotlib.pyplot as plt

n = 2

alpha_c = math.sqrt( (math.sqrt(4 + 9*n*n) - 3*n)/(math.sqrt(4 + 9*n*n) + 3*n) )
Cn2 = (math.sqrt(4 + 9*n*n) + 3*n)**3 / (128*n**3) * math.exp(-2*alpha_c/n )
r_c = (3*n*n+math.sqrt(4*n*n+9*n**4))/(4*n*n)


def eq39(alpha):
  return -alpha**2 + 1 - 2/rs1 + Cn2* math.exp(2*alpha/n)/rs1**4

def sol39(rs):
  global rs1
  rs1 = rs
  if rs<r_c:
     return scipy.optimize.brentq(eq39, -10, alpha_c)

  return scipy.optimize.brentq(eq39, alpha_c,1)

print(  sol39(3.) )
```

References

[1] Misner, Throne, Wheeler, Gravitation, Princeton, 1973.

[2] M. Hannam, S. Husa, F. Ohme, B. Brügmann, N. Ó Murchadha, Phys. Rev. **D 78**, 064020 (arxiv.org/abs/0804.0628).